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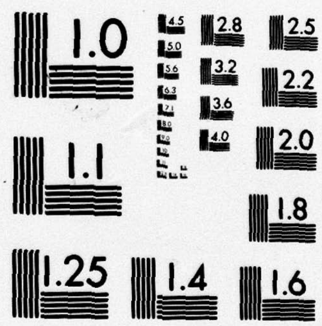

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DERIVATIVES AND CODIFFERENTIALS OF MAPS WITH CLOSED CONVEX GRAPHS AND CONVEX OPERATORS.

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DERIVATIVES AND CODIFFERENTIALS OF MAPS WITH CLOSED CONVEX GRAPHS AND CONVEX OPERATORS

Jean-Pierre Aubin

Technical Summary Report #2011
October 1979

ABSTRACT

We complete the study of contingent cones to a subset K and of the contingent derivatives of a set valued map F when K and the graph of F are closed and convex. In this case, the contingent cone is a closed convex cone (called the tangent cone) and the contingent derivative is a convex process (a set-valued map whose graph is a closed convex cone). The transpose of the contingent derivative is another convex process, called the codifferential, which plays a quite important role.

We present a calculus of tangent cones to closed convex sets and we adopt the basic results of convex analysis to the case of set-valued maps with closed convex graph.

This study is motivated by the crucial role played by convex cones in optimization, fixed-point theory and flow-invariance of dynamical systems.

AMS (MOS) Subject Classifications: 47M10, 47H15, 47H99, 49B30, 49B99.

Key Words: tangent cones, convex analysis, convex processes.

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SIGNIFICANCE AND EXPLANATION

In recent years, the concept of tangent cone to a closed convex subset K of a Hilbert space X played a crucial role in many fields. The tangent cone $T_K(x)$ to K at x is defined as the closure of the cone spanned by $K - x$. We mention for instance the Brouwer - Ky Fan theorem: If K is compact and convex and if f is a continuous map from K to X, satisfying: $\forall x \in K, f(x) \in T_K(x)$ there exists a solution $x_* \in K$ to the equation $f(x_*) = 0$.

Also, the Nagumo theorem states that under the same assumptions, for all $x_0 \in K$, there exists a solution $x(0)$ to the differential equation $x' = f(x)$, $x(0) = x_0$ such that, $\forall t \geq 0, x(t) \in K$.

Finally, if V is a real-valued convex differentiable function defined on a neighborhood of K, then x_0 achieves the minimum of V on K if and only if

$$\forall u \in T_K(x_0), \langle \operatorname{grad} V(x_0), u \rangle \geq 0 \quad .$$

These three typical (and fundamental) results motivate a systematic study of tangent cones, which completes the study of the contingent cones presented in Aubin [5]. We also adapt the main results of convex analysis to set-valued maps (or multi functions) F (associating to $x \in X$ a subset $F(x)$, rather than to a point) that are convex in the sense that, for all $\alpha \in [0,1]$,

$$\alpha F(x) + (1-\alpha)F(y) \subset F(\alpha x + (1-\alpha)y) \quad .$$

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the author of this report.

DERIVATIVES AND CODIFFERENTIALS OF MAPS
WITH CLOSED CONVEX GRAPHS AND CONVEX OPERATORS

Jean-Pierre Aubin

Introduction

When K is a closed convex subset of Hilbert space X, the contingent cone $D_K(x)$ to K at $x \in K$ coincides with the tangent cone $T_K(x)$ to K at x, defined by

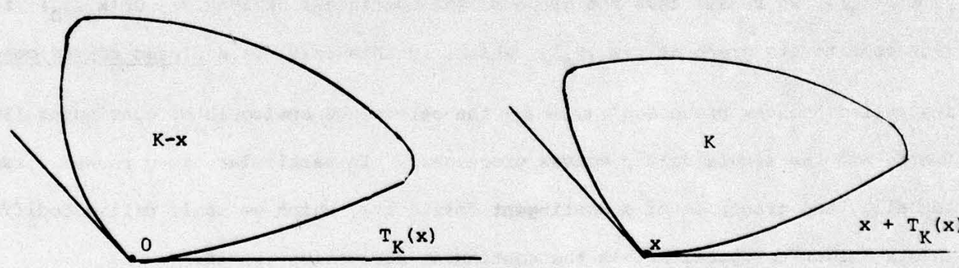
$$T_K(x) \doteq c\ell\left(\bigcup_{h>0} \frac{1}{h}(K-x)\right) .$$

It is the closure of the convex cone $S_K(x)$ defined by

$$S_K(x) \doteq \bigcup_{h>0} \frac{1}{h}(K-x)$$

spanned by $K - x$.

Hence $T_K(x)$ is always a convex cone, equal to X when $x \in \text{Int}(K)$.



We always have

$$K \subset x + T_K(x) .$$

By convention, we set

$$T_\emptyset(x) \doteq \emptyset .$$

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The first result states that when K is closed and convex, the tangent cone $T_K(x)$ coincides with the Bouligand contingent cone $D_K(x) = \bigcap_{h,\alpha>0} \bigcup_{h\in]0,\alpha]} (\frac{1}{h}(K-x) + \varepsilon B)$ (see Aubin [5], for instance). But, whereas we had only a poor calculus on contingent cones, consisting mostly of trivial inclusions or inequalities, the convexity of K allows to have a reasonable calculus, where equalities hold.

So, Haddad's theorem (see Haddad [1]) states that the property

$$\forall\, x \in K,\ F(x) \cap T_K(x) \neq \emptyset$$

is a necessary and sufficient condition for the differential inclusion $x'(t) \in F(x(t))$ to have viable trajectories, i.e., trajectories satisfying $x(t) \in K$ for all t. We also know that when K is compact and convex, this condition implies the existence of stationary points $\bar{x} \in K$, i.e., solutions to the inclusion $0 \in F(\bar{x})$ (see for instance Aubin [2], chap. 15). These two results only motivate a further study of these tangent cones.

As a first application, we complete the calculus of the contingent derivatives of a set-valued map with closed graph; when x_0 belongs to the domain of a set-valued map F and $y_0 \in F(x_0)$, we recall that the graph of the contingent derivative $DF(x_0,y_0)$ is the contingent cone to its graph at (x_0,y_0), which, in this case, is a closed convex cone. Such maps are called "convex processes"; they are the set-valued analogues of continuous linear maps (which are the single valued convex processes). In particular, they posses "transposes". Actually, the transpose of a contingent derivative, which we shall call a codifferential, enjoys "nicer" properties than the contingent derivative itself.

Let P be a closed convex cone of a Hilbert space Y, representing the set of "non-negative elements of Y". A convex operator A from X to Y is a single valued map defined on its domain $D(A) \subset X$ to Y that satisfies $\forall\, x, y \in D(A)$, $\forall\, \theta \in [0,1]$, $\theta A(x) + (1-\theta)A(y) \in A(\theta x + (1-\theta)y) + P$. Since A is convex if and only if the associated set valued map A_+ defined by $A_+(x) \doteq A(x) + P$ has a convex graph, we are able to define "upper derivatives and codifferentials" of a convex operator via the derivatives and codifferentials of the set-valued map A_+. When $Y = \mathbb{R}_+$, convex operators are convex functions and we derive the main results of convex analysis (see for instance Rockafellar [1], [2]).

Since the tangent cone $T_K(x)$ is a closed convex cone, it is quite useful to introduce the (negative) polar cone of a tangent cone: this is the normal cone to K at x, denoted by $N_K(x)$, which coincides with the subdifferential $\partial\psi_K(x)$ of the indicator of K at x:

$$N_K(x) = \{p \in X \mid \langle p,x\rangle = \max_{y\in X} \langle p,y\rangle\} \ .$$

When X is finite dimensional, we shall prove that the set valued map $x \mapsto T_K(x)$ is lower semicontinuous, that the graph of $x \mapsto N_K(x)$ is closed and, when $\mathrm{Int}\, K \neq \emptyset$, that the graph of $x \mapsto \mathrm{Int}\, T_K(x) = \bigcup_{h>0} \frac{1}{h}(\mathrm{Int}\, K - x)$ is open. Finally, we provide several formulas. If $K = B$, is the unit ball, then

$$\text{if} \quad \|x\| = 1, \ T_B(x) = \{x\}^- \ .$$

When K is a closed convex cone, then;

$$\begin{cases} v \in T_K(x) \ \text{if and only if} \ \forall\, p \in K^- \ \text{satisfying} \ \langle p,x\rangle = 0 \ , \\ \text{we have} \ \langle p,v\rangle \leq 0 \ . \end{cases}$$

When $K = \mathbb{R}^n_+$ is the positive orthant, this formula becomes:

$$v \in T_{\mathbb{R}^n_+}(x) \ \text{if and only if} \ v_i \geq 0 \ \text{whenever} \ x_i = 0 \ .$$

If $K = M^n \doteq \{x \in \mathbb{R}^n_+ \mid \sum x_i = 1\}$, we obtain

$$v \in T_{M^n}(x) \ \textit{if and only if} \ \sum_{i=1}^n v_i = 0 \ \text{and} \ v_i \geq 0 \ \text{whenever} \ x_i = 0 \ .$$

When $A \in \mathcal{L}(X,Y)$ and $y \in Y$ are given, we check that if $Ax = y$,

$$T_{A^{-1}(y)}(x) = \mathrm{Ker}\, A \ .$$

We shall prove also the following formulas:

$$T_{\prod_{i=1}^n K_i}(x_1,\ldots,x_n) = \prod_{i=1}^n T_{K_i}(x_i)$$

if $A \in \mathcal{L}(X,Y)$,

$$T_{\overline{A(K)}}(Ax) = \overline{AT_K(x)} \quad .$$

If $L \subset X$ and $M \subset Y$ are closed convex subsets satisfying the condition $0 \in \text{Int}(A(L) - M)$, then if $x \in L \cap A^{-1}(M)$,

$$T_{L \cap A^{-1}(M)}(x) = T_L(x) \cap A^{-1}(T_M(Ax)) \quad .$$

The latter formula will play a very important role in applications. In particular, if $L \subset X$, $M \subset X$ satisfy $0 \in \text{Int}(L-M)$, than we obtain

$$T_{L \cap M}(x) = T_L(x) \cap T_M(x) \quad .$$

We consider then the contingent derivative $DF(x_0,y_0)$ of a map F from X to Y with closed convex graph: its graph is a closed convex cone; A map whose graph is a closed convex cone is called a "convex process". These maps are the set-valued analogues of continuous linear operators (which are single valued convex processes). One can hope that the contingent derivatives of a map with closed convex graph, inheriting the properties of tangent cones to closed convex subsets, enjoy a "nice" calculus. This is not exactly the situation.

Fortunately, we can define the transpose G^* of a convex process G from X to Y, which is a convex process from Y^* to X^*. It happens that it is the transpose $DF(x_0,y_0)^*$ of the contingent derivative $DF(x_0,y_0)$ that benefits of most of the expected properties. We call it the codifferential of F at (x_0,y_0).

Also, the graph of F is characterized by its support function, which, up to a minus sign, is equal to

$$f^*(p,q) = \sup_{x \in X} \sup_{y \in F(x)} [\langle p,x \rangle - \langle q,y \rangle] \quad .$$

This function shall play a quite important role. We call it the conjugate function of F.

Actually, these concepts are consistent with the familiar concepts of convex analysis. If V is a lower semicontinuous convex function from $K \subset X$ to $\mathbb{R}$. The set-valued map V_+ from K to $\mathbb{R}$ defined by

$$V_+(x) \doteq V(x) + \mathbb{R}_+$$

has a closed convex graph.

We shall see that

$$DV_+(x,V(x))(u) = [D_+V(x)(u),\infty[$$

where

$$D_+V(x)(u) = \liminf_{v\to u} \inf_{h>0} \frac{V(x+hv) - V(x)}{h} ,$$

that, if $q = 1 \in \mathbb{R}(=Y^*)$,

$$\forall p \in X^*, \ (V_+)^*(p,1) = V^*(p)$$

defines the conjugate function of V and that

$$DV_+(x,V(x))^*(1) = \partial V(x)$$

is the subdifferential of V at x.

So, the duality between directional derivative $D_+V(x)$ and subdifferentials $\partial V(x)$ is preserved in the general framework.

By the way, we shall define "convex operators" A from X to a Hilbert space ordered by a closed convex cone P to be the single valued maps satisfying, for all $\alpha \in [0,1]$,

$$\alpha A(x) + (1-\alpha)A(y) \in A(\alpha x + (1-\alpha)y) + P \quad .$$

Since the graph of the set-valued map A_+ defined by

$$A_+(x) \doteq A(x) + P$$

is convex, these maps will enjoy the properties of the set-valued maps with closed convex graph which we are about to summarize.

We begin with the following characterizations of the codifferential $DF(x_0y_0)^*$: the statements below are equivalent:

a). $p_0 \in DF(x_0,y_0)^*(q_0)$

b). $\forall\, x \in X,\ \forall\, y \in F(x),\ \langle q_0, y_0 - y\rangle \le \langle p_0, x_0 - x\rangle$

c). $\langle p_0, x_0\rangle = f^*(p_0,q_0) + \langle q_0, y_0\rangle$

d). $\forall\, u \in X,\ \forall\, v \in DF(x_0,y_0)(u),\ \langle p_0, v\rangle \le \langle q_0, u\rangle$.

The variational principle holds: When $P \subset Y$ is a closed convex cone, then $(x_0,y_0) \in \text{graph}\,(F)$ achieves the minimum of F in the sense that

$$\forall\, x \in X,\ F(x) \subset y_0 + P$$

if and only if

$$\forall\, u \in X,\ DF(x_0,y_0)(u) \subset P \ .$$

The expected formulas hold.

a) If $A \in \mathcal{L}(Z,X)$ and if $0 \in \text{Int}\,(\text{Im}\,A - \text{Dom}\,F)$, then the chain rule

$$D(FA)(z_0,y_0)^* = A^* DF(Az_0,y_0)^*$$

holds.

b) If $A \in \mathcal{L}(Y,X)$ and if $\overline{AF}$ denotes the map whose graph is the closure of the graph of AF, then

$$D(\overline{AF})(x_0,Ay_0)^* = DF(x_0,y_0)^* A^* \ .$$

c) If F_1 is an upper hemicontinuous map with compact values and convex graph, if F_2 is a map with closed convex graph and if

$$0 \in \text{Int}\,(\text{Dom}\,F_1 - \text{Dom}\,F_2)$$

then

$$D(F_1 + F_2)(x_0,\ y_1 + y_2)^* = DF_1(x_0,y_1)^* + DF_2(x_0,y_2)^* \ .$$

d) Let K be a subset of X and $F|_K$ denote the restriction of F to K. If $0 \in \text{Int}\,(K - \text{Dom}\,F)$, then

$$D(F|_K)(x_0,y_0)^* = DF(x_0,y_0)^* + N_K(x_0) \ .$$

Outline

1. Tangent cones.

Definition 1

Let $K \subset X$ be a closed convex subset and $x \in K$. We denote by

$$S_K(x) \doteq \bigcup_{h>0} \frac{1}{h}(K-x) \tag{1}$$

the cone spanned by $K - x$ and by

$$T_K(x) \doteq c\ell\Big(\bigcup_{h>0} \frac{1}{h}(K-x)\Big) \tag{2}$$

its closure. $T_K(x)$ is called the "tangent cone to K at x". First, we note that $v \in S_K(x)$ if and only if $x + hv \in K$ for some $h > 0$ and that $v \in T_K(x)$ if and only if there exist a sequence of elements v_n converging to v and positive h_n such that $x + h_n v_n \in K$ for all $n > 0$.

Also, if $v \in S_K(x)$ and thus, if $x + hv \in K$ where $h \geq 0$, then, for all $k \in [0,h]$, $x + kv \in K$: Indeed, $x + kv = (1 - \frac{k}{h})x + \frac{k}{h}(x + hv)$ is a convex combination of elements belonging to the convex set K.

Also, the cone $S_K(v)$ is obviously convex: Indeed, if v_1 and v_2 belong to $S_K(v)$, then $x + h_i v_i \in K$ for $i = 1,2$; let $h = \min(h_1,h_2)$. Then, by the preceding remark, $x + hv_i \in K$ for $i = 1,2$. Hence $x + h(\alpha v_1 + (1-\alpha)v_2) \in K$ when $\alpha \in [0,1]$. Since the closure of a convex cone is still a convex cone, we have proved the following proposition.

Proposition 1

The cones $S_K(x)$ and $T_K(x)$ are convex. ▲

It is also clear that

$$K \subset x + S_K(x) \subset x + T_K(x) \tag{3}$$

and that

$$\text{if } x \in \mathrm{Int}\, K, \text{ then } T_K(x) = X \tag{4}$$

since $K - x$ contains a neighborhood of the origin.

The first property is that the tangent cone $T_K(x)$ coincides with the contingent cone $D_K(x)$, defined by

(5) $$D_K(x) = \bigcap_{\varepsilon,\alpha>0} \bigcup_{h\in]0,\alpha]} \left(\frac{1}{h}(K-x) + \varepsilon B\right) .$$

(See Aubin [5], for instance.) It is obvious that we always have

$$D_K(x) \subset T_K(x) .$$

Theorem 1

When K is closed and convex, then

(6) $$\forall\, x \in K,\ T_K(x) = D_K(x) .$$

Proof:

Let $v \in T_K(x) \doteq c\ell(S_K(x))$. Then, for all $\varepsilon > 0$, there exist $y_\varepsilon \in K$ and $h_\varepsilon > 0$ such that

$$v - \frac{1}{h_\varepsilon}(y_\varepsilon - x) \in \varepsilon B$$

i.e., such that $x + h_\varepsilon v_\varepsilon \in K$ where $v_\varepsilon = \frac{1}{h_\varepsilon}(y_\varepsilon - x)$.

Let us associate with any $\alpha > 0$ the positive number $h = \min(\alpha, h_\varepsilon) > 0$. Since K is convex, $x + h\left(\frac{y_\varepsilon - x)}{h_\varepsilon}\right) = (1 - \frac{h}{h_\varepsilon})x + \frac{h}{h_\varepsilon}(x + h_\varepsilon v_\varepsilon)$ belongs to K. This proves that for any $\varepsilon > 0$ and $\alpha > 0$, there exist $h \leq \alpha$ and $v_\varepsilon \in v + \varepsilon B$ such that $x + hv_\varepsilon \in K$, i.e. that $v \in D_K(x)$. ∎

2. Normal cones.

We recall the definition of normal cones to a closed convex subset.

Definition 1

Let K be a nonempty closed convex subset of X. The "normal cone" $N_K(x)$ to K at x is defined by

(1) $$N_K(x) = \{p \in X^* \text{ such that } \langle p,x\rangle = \max\{\langle p,y\rangle \mid y \in K\}\} .$$

Proposition 1

The normal cone $N_K(x)$ is the (negative) polar cone $T_K(x)^-$ of the tangent cone $T_K(x)$ ▲

Therefore, $T_K(x) = N_K(x)^-$ since $T_K(x)$ is a closed convex cone.

Proof: a) If $p \in T_K(x)^-$, then $\langle p, y - x\rangle \leq 0$ for all $y \in K$ since $v = y - x \in T_K(x)$ when $y \in K$. Hence $p \in N_K(x)$.

b) Conversely, let $p \in N_K(x)$ and $v = \lim\limits_{n\to\infty} \lambda_n(y_n - x) \in T_k(x)$, where $\lambda_n \geq 0$ and $y_n \in K$. Hence $\langle p, v\rangle \leq 0$ since $\langle p, \lambda_n(y_n - x)\rangle = \lambda_n \langle p, y_n - x\rangle \leq 0$ for all $n \geq 0$. ■

We also recall the following property:

Proposition 2

The normal cone $N_K(x)$ is the subdifferential $\partial\phi_K(x)$ of the indicator function $\phi_K(\cdot)$ of K. ▲

Proof: $p \in \partial\phi_K(x)$ if and only if $\phi_K(x) - \phi_K(y) \leq \langle p, x - y\rangle$ for all $y \in K$, i.e., if and only if $p \in N_K(x)$.

Proposition 3

Let π_K be the projection of best approximation onto K. Then $\pi_K^{-1}(x) = x + N_K(x)$ and $v \in T_K(x)$ if and only if $\langle y - x, v\rangle \leq 0 \ \forall y \in \pi_K^{-1}(x)$. ▲

Proof: a) If $p \in N_K(x)$, then $x = \pi_K(x+p)$ since $\langle (x+p) - x, y - x\rangle = \langle p, y - x\rangle \leq 0$ for all $y \in K$.

b) Conversely, let $y \in \pi_K^{-1}(x)$; then

$$\langle y - x, z - x \rangle \leq 0 \quad \text{for all} \quad z \in K \ .$$

Hence $p = y - x \in N_K(x)$.

c) The last statement follows from the first and from the fact that $T_K(x) = N_K(x)^-$.

▲

3. Continuity properties of the tangent and normal cones.

Theorem 1

Let K be a closed convex subset of a finite dimensional space. Then

$$(1)\qquad \begin{cases} \text{i)} & x \mapsto N_K(x) \text{ has a closed graph} \\ \text{ii)} & x \mapsto T_K(x) \text{ is lower semicontinuous} \end{cases} \quad . \qquad \blacktriangle$$

Proof: a) Let (x_n,p_n) be a sequence of elements of the graph of $N(\cdot)$ converging to (x,p). For all $y \in K$, we have $\langle p_n,y\rangle \leq \langle p_n,x_n\rangle$. Hence, letting $n \to \infty$, we deduce that $\langle p,y\rangle \leq \langle p,x\rangle$. Thus (x,p) belongs to the graph of $N(\cdot)$.

b) The second statement is equivalent to the first by a result of Aubin [4]. ■

Proposition 1

If $\text{Int } K \neq \emptyset$, then $\text{Int } T_K(x) \neq \emptyset$ and is equal to the cone spanned by $\text{Int } K - x$:

$$(2)\qquad \text{Int } T_K(x) = \bigcup_{h>0} \frac{1}{h}(\text{Int } K - x) \quad .$$

Furthermore, the set valued map $x \mapsto \text{Int } T_K(x)$ has an open graph. ▲

Proof: a) The cone $\bigcup_{h>0} \frac{1}{h}(\text{Int } K - x)$, being a union of open subsets, is open. So, it is contained in $\text{Int } T_K(x)$. But its closure is equal to $T_K(x)$: Indeed

$$\bigcup_{h>0} \frac{1}{h}(K-x) = \bigcup_{h>0} \overline{\frac{1}{h}(\text{Int } K - x)} \subset \overline{\bigcup_{h>0} \frac{1}{h}(\text{Int } K - x)} \subset T_K(x) \quad .$$

Since $T_K(x)$ is convex, it is the closure of its interior. Hence formula (2) holds true.

b) Let $v_0 \in \text{Int } T_K(x_0)$. Therefore, $v_0 \in \frac{1}{h_0}(\text{Int } K - x_0)$ for some $h_0 > 0$; hence there exists $\varepsilon > 0$ such that $x_0 + h_0v_0 + \varepsilon B = x_0 + h_0(v_0 + \frac{\varepsilon}{h_0}B) \subset \text{Int } K$. Take $x \in x_0 + \varepsilon/2B$ and $v \in v_0 + \varepsilon/2h_0B$. Then $x + h_0v \in x_0 + h_0v_0 + \varepsilon B \subset \text{Int } K$ and therefore, $v \in \text{Int } T_K(x)$. Hence the graph of $x \to \text{Int } T_K(x)$ is open. ■

4. Tangent cones to some closed convex sets

Let B be the unit ball of a Hilbert space and $x \in B_0$. Then

$$(1) \quad \begin{cases} \text{i)} & T_B(x) = X \text{ if } x \in \operatorname{Int} B \text{ and } T_B(x) = \{x\}^- \text{ if } \|x\| = 1 . \\ \text{ii)} & N_B(x) = \{0\} \text{ if } x \in \operatorname{Int} B \text{ and } N_B(x) = \mathbb{R}_+ x \text{ if } \|x\| = 1 . \end{cases}$$ ▲

Proof: We take $\|x\| = 1$.

Then $p \in N_B(x)$ if and only if $\|p\|_* = \sup_{y \in B} \langle p,y \rangle = \langle p,x \rangle$. By the Cauchy-Schwarz inequality, this is equivalent to $p = \lambda x$ with $\lambda > 0$.

By polarity, we deduce the formula for the tangent cone. ■

Proposition 2

Let $K \subset X$ be a closed convex cone. Then $N_K(x) = K^- \cap \{x\}^\perp$ and thus:

$$(2) \quad \begin{cases} v \in T_K(x) \text{ if and only if } \langle p,v \rangle \leq 0 \text{ for all } p \in K^- \text{ satisfying} \\ \langle p,x \rangle = 0 . \end{cases}$$

If K is a closed subspace, then $T_K(x) = K$ and $N_K(x) = K^\perp$. ▲

Proof:

It is clear that $K^- \cap \{x\}^\perp$ is contained in $N_K(x)$. Conversely, if $p \in N_K(x)$, then $\langle p,x \rangle = \max_{y \in K} \langle p,y \rangle$. Since K is a cone, we deduce that $\langle p,x \rangle = 0$ and that $p \in K^-$. ■

Proposition 3

Let $A \in \mathcal{L}(X,Y)$ and $K = A^{-1}(y)$ be an affine subspace. Then, if $Ax = y$,

$$(3) \quad T_{A^{-1}(y)}(x) = \operatorname{Ker} A .$$

Proof: a) If $v \in \operatorname{Ker} A$, then $v + x \in A^{-1}(y) = K$ and thus, $v = v + x - x$ belongs to $T_K(x)$.

b) Conversely, if $v = \lim_{n\to\infty} v_n \in T_K(x)$, where $v_n = \lambda_n(x_n - x)$ with $x_n \in K$ and $\lambda_n > 0$, then $v_n \in \operatorname{Ker} A$ and thus $v \in \operatorname{Ker} A$. ■

Proposition 4

Let $M^n \doteq \{x \in \mathbb{R}^n_+ \mid \sum_{i=1}^n x_i = 1\}$. Let $I(x) \quad \{i=1,\dots,n \mid x_i = 0\}$. Then

(4) $$v \in T_{\mathbb{R}^n_+}(x) \text{ if and only if } v_i \geq 0 \text{ for all } i \in I(x)$$

and

(5) $$\begin{cases} v \in T_{M^n}(x) \text{ if and only if } v_i \geq 0 \text{ for all } i \in I(x) \\ \text{and } \sum_{i=1}^{n} v_i = 0 \ . \end{cases}$$

Proof: a) If $K = \mathbb{R}^n_+$, the first statement follows from Proposition 3: If $p \in \mathbb{R}^n_-$ satisfies $\sum_{i=1}^{n} p_i x_i = \sum_{i \notin I(x)} p_i x_i = 0$, then $p_i = 0$ whenever $i \notin I(x)$; hence $v \in T_{\mathbb{R}^n_+}(x)$ if $\sum_{i \in I(x)} p_i v_i \leq 0$ for all $p \in \mathbb{R}^n_-$, i.e., if and only if (4) holds.

b) Let v satisfy $v_i \geq 0$ if $i \in I(x)$ and $\sum_{i=1}^{n} v_i = 0$. If $v_i = 0$ for all $i \notin I(x)$, then $v = 0$. If not, let $\lambda = \min\limits_{\substack{i \notin I(x) \\ v_i = 0}} \frac{x_i}{|v_i|} > 0$. Therefore $x + \lambda v \in M^n$ since $x_i + \lambda v_i = \lambda_i v_i \geq 0$ if $i \in I(x)$, $x_i + \lambda v_i \geq x_i - \lambda|v_i| \geq x_i - x_i = 0$ if $i \notin I(x)$ and $\sum_{i=1}^{n} (x_i + \lambda v_i) = 1 + 0 = 1$. Hence $v \in \frac{1}{\lambda}(M^n - x) \in T_{M_n}(x)$.

c) If $v = \lambda(y-x)$ where $y \in M^n$ and $\lambda > 0$, then we deduce that $v_i = \lambda(y_i - x_i) = \lambda y_i \geq 0$ when $i \in I(x)$ and that $\sum_{i=1}^{n} v_i = 0$. Therefore $\bigcup_{\lambda>0} \lambda(M^n - x) \subset \{v \in T_{\mathbb{R}^n_+}(x) \mid \sum_{i=1}^{n} v_i\} = 0$. Since the latter subset is closed, we deduce that $T_{M^n}(x) = \{v \in T_{\mathbb{R}^n_+}(x) \mid \sum_{i=1}^{n} v_i = 0\}$. ■

5. Calculus on tangent cones to closed convex subsets

Since the tangent cones to a closed convex subset $K \subset X$ coincide with the contingent cones, the tangent cones inherit the properties of the contingent cones. They do enjoy other properties that are quite useful. We begin by recalling the obvious properties.

We assume once and for all that all the subsets that are involved in the following statements are nonempty, closed and convex.

Proposition 1

a) If $x \in K \subset L$, then

(1) $$T_K(x) \subset T_L(x) \quad \text{and} \quad N_L(x) \subset N_K(x).$$

b) Let $K \doteq \bigcap_{i \in J} K_i$, and $J(x) = \{i \mid x_i \notin \text{Int } K_i\}$. Then

(2) $$T_K(x) \subset \bigcap_{i \in J(x)} T_{K_i}(x).$$

c) Let $K \doteq \bigcup_{i \in I} K_i$ and $I(x) \quad \{i \in I \mid x \in K_i\}$. Then

(3) $$\bigcup_{i \in I(x)} T_{K_i}(x) \subset T_K(x) \quad .$$

Equality holds when I is finite or, more generally, locally finite. ▲

(See Aubin [5], propositions 2.1, 2.2 and 2.3.)

Proposition 2

Let $\vec{K} \doteq \prod_{i=1}^{n} K_i$ and $\vec{x} = (x_1, \ldots, x_n) \in \vec{K}$. Then

(4) $$T_{\vec{K}}(\vec{x}) = \prod_{i=1}^{n} T_{K_i}(x_i) \quad \text{and} \quad N_{\vec{K}}(\vec{x}) = \prod_{i=1}^{n} N_{K_i}(x_i) \; .$$

Proof:

It is obvious that $T_{\vec{K}}(\vec{x}) \subset \prod_{i=1}^{n} T_{K_i}(x_i)$. Conversely, let $v_i \in T_{K_i}(x_i)$ for $i=1,..,n$.

Then there exist sequences of elements v_i^k converging to v_i and of $h_i^k > 0$ such that $x_i + h_i^k v_i^k \in K_i$ for all $i=1,..,n$. We set $h^k = \min_{i=1,..,n} h_i^k > 0$. Since the subsets K_i are convex, $x_i + h^k v_i^k \in K_i$ for all, i.e., $\vec{x} + h\vec{v}^k \in \vec{K}$. Hence $\vec{v} \in T_{\vec{K}}(\vec{x})$. We deduce the formula on normal cones by polarity. ■

Proposition 3

Let $A \in \mathcal{L}(X,Y)$ and $K \subset X$. Then,

(5) $\quad \forall x \in K, \ T_{\overline{A(K)}}(Ax) = c\ell(AT_K(x))$

and, by polarity,

(6) $\quad N_{\overline{A(K)}}(x) = A^{*-1}(N_K(x))$. ▲

Proof:

Since $\langle p, Ax\rangle = \max_{y\in K} \langle p, Ay\rangle = \max_{y\in K} \langle A^* p, y\rangle = \langle A^* p, x\rangle$, we obtain the formula for the normal cones and deduce it by polarity for tangent cones. ■

Corollary 1

Let K and L be two closed convex subsets, $x \in K$ and $y \in L$. Then

(7) $\quad T_{\overline{K+L}}(x+y) = c\ell(T_K(x) + T_L(y))$ and $N_{\overline{K+L}}(x+y) = N_K(x) \cap N_L(y)$. ▲

Theorem 1

Let $A \in \mathcal{L}(X,Y)$, $L \subset X$ and $M \subset Y$. We set

(8) $\quad K \doteq \{x \in L \mid Ax \in M\} = L \cap A^{-1}(M)$.

Assume that $K \neq \emptyset$, i.e., that $0 \in A(L) - M$. The inclusion

(9) $\quad T_K(x) \subset T_L(x) \cap A^{-1}(T_M(Ax))$

is always true. If we assume that

(10) $\quad 0 \in \mathrm{Int}(A(L) - M)$,

then the equality

(11) $\quad T_K(x) = T_L(x) \cap A^{-1}(T_M(Ax))$

holds true. ▲

Proof:

The first inclusion is obvious. The equality shall follow from the Robinson - Ursescu theorem. (See Robinson [1], Ursescu [1].) Let $x_0 \in K$ and $v_0 \in T_L(x_0) \cap A^{-1}T_M(Ax_0)$. There

exist sequences of elements $v_n \in X$ and $u_n \in Y$ converging to v_0 and Av_0 respectively such that, for all n, $x_0 + h_n^1 v_n \in L$ and $Ax_0 + h_n^2 u_n \in M$. We set $h_n \doteq \min(h_n^1, h_n^2, 1) > 0$. Since L and M are <u>convex</u>, we deduce that

(12) for all n, $x_n \doteq x_0 + h_n v_n \in L$ and $y_n \doteq Ax_0 + h_n u_n \in M$.

The theorem is proved if $u_n = Av_n$ for an infinite subset of indexes. If not, we apply the Robinson - Ursescu theorem to the set-valued map F defined from L to Y by

(13) $F(x) \doteq Ax \in M$.

We take $y_0 = 0$ and $x_0 \in F^{-1}(0) = K$. By assumption (10), $y_0 \in \mathrm{Int}\, F(L) = \mathrm{Int}(A(L) - K)$. Hence, there exists $\gamma > 0$ such that

(14) $\forall x \in L,\ d(x, F^{-1}(y)) \leq \frac{1}{\gamma} d(y, F(x))(\|x_0 - x\| + 1)$.

We take $y = 0$ and $x = x_0 + h_n v_0$. So $\|x_0 - x\| = h_n\|v_0\|$, $d(0, F(x_0 + h_n v_0)) = d(Ax_0 + h_n Av_0, M) \leq d(Ax_0 + h_n u_n, M) + h_n\|Av_0 - u_n\| = h_n\|Av_0 - u_n\|$. Therefore,

(15) $$\frac{d(x_0 + h_n v_0, K)}{h_n} = \frac{1}{h_n} d(x_0 + h_n v_0, F^{-1}(y_0)) \leq \frac{1}{\gamma h_n} d(0, F(x_0 + h_n v_0))(\|x - x_0\| + 1)$$
$$\leq \frac{1}{\gamma} \|Av_0 - u_n\|(h_n\|v_0\| + 1) .$$

Hence

(16) $$\inf_{h_n > 0} \frac{d(x_0 + h_n v_0, K)}{h_n} = 0 .$$

This means that $v_0 \in T_K(x_0)$.

By polarity, we deduce that assumption (10) implies

(17) $$N_K(x) = c\ell(N_L(x) + A^* N_M(Ax)) .$$ ■

It also implies that $N_L(x) + A^* N_M(Ax)$ <u>is closed</u>. Actually, it implies that the map

(18) $$(p,q) \in N_L(x) \times N_M(Ax) \mapsto p + A^* q \in X^*$$

is "proper". This means that

$$(19)\quad \begin{cases} \text{If the sequence } p_n + A^* q_n \text{ (where } p_n \in N_L(x) \text{ and } q_n \in N_M(Ax)) \\ \text{converges strongly in } X^*, \text{ then subsequences of } p_n \text{ and } q_n \\ \text{converge weakly in } X^* \text{ and } Y^* \text{ respectively.} \end{cases}$$

(See Aubin [1], chap. 3, §4, p. 93).

Proposition 4.

Let X, Y be Hilbert spaces, $A \in \mathcal{L}(X,Y)$, $L \subset X$, $M \subset Y$ satisfy

$$(20)\qquad 0 \in \operatorname{Int}(A(L) - M) \quad .$$

Then the map defined by (18) is proper and, consequently:

$$(21)\qquad \forall\, x \in K,\ N_K(x) = N_L(x) + A^* N_M(Ax) \quad . \qquad \blacktriangle$$

Proof:

Let $r_n = p_n + A^* q_n$ be a sequence converging to r, where $p_n \in N_L(x)$ and $q_n \in N_M(Ax)$. We prove that

$$(22)\qquad \forall\, v \in Y,\ \sup_n \langle q_n, v\rangle < +\infty \quad .$$

Indeed, there exist $\lambda > 0$, $y \in L$ and $z \in M$ such that $v = \lambda(z - Ay)$ by assumption (20). Hence

$$\langle q_n, v\rangle = \lambda\langle q_n, z - Ay\rangle = \lambda(\langle q_n, z\rangle - \langle A^* q_n, y\rangle)$$

$$= \lambda(\langle p_n, y\rangle + \langle q_n, z\rangle - \langle r_n, y\rangle)$$

$$\leq \ (\langle p_n, x\rangle + \langle q_n, Ax\rangle - \langle r_n, y\rangle)$$

$$(\text{since } p_n \in N_L(x) \text{ and } q_n \in N_M(Ax))$$

$$= \lambda\langle r_n, x - y\rangle \leq \lambda \|r_n\|\ \|x - y\| < +\infty$$

since the converging sequence r_n is bounded.

By the uniform boundedness theorem (see Aubin [2], chap. 4, §1, p. 73 for instance) the sequence of elements q_n is bounded. Therefore, it is weakly relatively compact and thus,

some subsequence (again denoted by) q_n converges to $q \in N_M(Ax)$. Hence $p_n = r_n - A^* q_n$ converges to $p = r - A^* q \in N_L(x)$ and thus, $r = p + A^* q$ belongs to $N_L(x) + A^* N_M(x)$. ∎

Corollary 2

If L is a closed convex subset of X and if $A \in \mathcal{L}(X,Y)$ then, for any $y \in \text{Int}\, A(L)$ and $x \in L \cap A^{-1}(y)$, we have

$$(23) \qquad T_{L \cap A^{-1}(y)}(x) = T_L(x) \cap \ker A \quad . \qquad \blacktriangle$$

Corollary 3

Let $P \subset X$ be a closed convex cone and $p_0 \in P^-$ such that $K = \{x \in P \text{ such that } \langle p_0, x\rangle = -1\}$ is not empty. Then

$$(24) \qquad \begin{cases} v \in T_K(x) \text{ if and only if } \langle p_0, v\rangle = 0 \text{ and } \langle p, v\rangle \le 0 \\ \text{for all } p \in P^- \text{ satisfying } \langle p, x\rangle = 0 \end{cases}$$

and

$$(25) \qquad N_K(x) = \ker p_0 + (P^- \cap \{x\}^\perp) \quad . \qquad \blacktriangle$$

By taking $X = Y$ and A to be the identity, we obtain:

Corollary 4

Let K and L be two closed convex subsets of X. If

$$(26) \qquad 0 \in \text{Int}(K-L)$$

than

$$(27) \qquad \forall x \in K \cap L,\ T_{K\cap L}(x) = T_K(x) \cap T_L(x) \quad . \qquad \blacktriangle$$

The equality between the tangent cone of a finite intersection and the intersection of tangent cones hold true under the following assumptions.

Proposition 5

Let $K \doteq \bigcap_{i=0}^{n} K_i$ be the intersection of $n + 1$ closed convex subsets. Let us assume that

$$(28) \qquad \begin{cases} \exists \gamma > 0 \text{ such that, for all } v_i \in \gamma B (i=1,..,n), \text{ there exists} \\ \quad x_0 \in K_0 \text{ such that } x_0 + v_i \in K_i \text{ for } i = 1,\dots,n \quad . \end{cases}$$

Then

(29) $$\forall\, x \in K,\ T_K(x) = \bigcap_{i=0}^{n} T_{K_i}(x) \quad .$$

<u>Proof</u>:

We consider the map A defined from X^{n+1} to X^n by

(30) $$A(x_0,x_1,\ldots,x_n) = (x_0 - x_1,\ x_0 - x_2,\ldots,x_0 - x_n) \quad .$$

We set $L = K_0 \times K_1 \times\cdots\times K_n \subset X^{n+1}$ and $M = \{0\}$. So $K = L \cap A^{-1}(M)$. Assumption (28) amounts to saying that $0 \in \operatorname{Int}(A(L) - M)$. Therefore, when $x \in K$ and when $\vec{x} \doteq (x,\ldots,x) \in L$, then $T_L(\vec{x}) = \prod_{i=0}^{n} T_{K_i}(x)$ and $T_M(0) = 0$. So $\vec{v} \in T_K(x)$ if and only if $v_i \in T_{K_i}(x)$ for all $i = 0,\ldots,n$ and $v_0 - v_1 = 0,\ldots,v_0 - v_n = 0$ $(i=1,\ldots,n)$. This means that $v_0 \in \bigcap_{i=0}^{n} T_{K_i}(x)$. ∎

6. Derivatives and codifferentials of set valued maps with closed convex graph.

We shall complete the analysis of the properties of contingent derivatives in the case when the set valued map F from X to Y has a closed convex graph.

Recall that the graph of F is convex if and only if for any convex combination $\sum_{i=1}^{n} \lambda^i x_i$ of elements $x_i \in X$, we have

$$\sum_{i=1}^{n} \lambda^i F(x_i) \subset F\left(\sum_{i=1}^{n} \lambda^i x_i\right) .$$

We shall see that convex functions and, more generally, convex operators yield examples of set-valued maps with convex graph.

We also recall that upper hemicontinuous set-valued maps from X to the closed convex subsets of Y have a closed graph.

Contingent derivatives of set-valued maps with closed convex graphs enjoy the properties of tangent cones to closed convex subsets.

Proposition 1

Let F be a proper set-valued map from $K \subset X$ to Y. Then the contingent derivatives $DF(x,y)$ are convex processes, i.e., set-valued maps whose graphs are closed convex cones. ▲

Proof:

Indeed, the graph of $DF(x,y)$, being the contingent cone to the graph of F at (x,y), is actually the tangent cone $T_{\text{graph}(F)}(x,y)$, which is convex. ∎

Since $D_{\text{graph}(F)}(x,y) = T_{\text{graph}(F)}(x,y)$, the formula of the contingent derivative may be simplified. We obtain:

$$\text{(1)} \qquad v_0 \in DF(x_0,y_0)(u_0) \iff \liminf_{u \to u_0} \lim_{h \to 0+} d\left(v_0, \frac{F(x_0 + hu) - y_0}{h}\right) = 0 .$$

If F is locally Lipschitzean, we obtain

$$\text{(2)} \qquad v_0 \in DF(x_0,y_0)(u_0) \iff \lim_{h \to 0+} d\left(v_0, \frac{F(x_0 + hu_0) - y_0}{h}\right) = 0 .$$

Definition 1

For simplicity, when the graph of F is closed and convex, we shall say that $DF(x_0,y_0)$ is the "derivative" of F at (x_0,y_0).

The convexity of the graph of F implies that for all $y \in F(x)$, when $h_1 \leq h_2$, then $\frac{F(x + h_2u) - y}{\theta_2} \subset \frac{F(x + h_1u) - y}{\theta_1}$. Indeed, for all $y \in F(x)$, we can write

$\frac{h_1}{h_2} F(x + h_2v) + (1 - \frac{h_1}{h_2})y \subset F(\frac{h_1}{h_2} (x + h_2v) + (1 - \frac{h_1}{h_2})x) = F(x + h_1v)$. This amounts to saying that the function $\theta \mapsto d(v, \frac{F(x+\theta u) - y}{\theta})$ is increasing. So, we can write

$$\text{(3)} \qquad \lim_{h\to 0+} d(v_0, \frac{F(x_0 + hu) - y_0}{h}) = \inf_{h>0} d(v_0, \frac{F(x_0 + hu) - y_0}{h}) \quad .$$

We recall that

$$DF(x_0,y_0)^{-1} = D(F^{-1})(y_0,x_0) \quad \text{for all} \quad x_0 \in K,\ y_0 \in F(x_0) \quad .$$

Remark.

Let $y_0 \in F(x_0)$. Since the images $F(x)$ are closed and convex, we can form the differential quotients

$$\text{(4)} \qquad \frac{1}{h} (\pi_{F(x_0 + hu_0)} y_0 - y_0) \quad .$$

If the limit when $h \to 0+$ of these differential quotients does exist, we shall say that F has a directional derivative at (x_0,y_0) in the direction u_0. We shall set

$$\text{(5)} \qquad \nabla F(x_0,y_0)(u_0) \doteq \lim_{h\to 0+} \frac{1}{h} (\pi_{F(x_0 + hu_0)} y_0 - y_0) \quad .$$

It is obvious that in this case

$$\text{(6)} \qquad \nabla F(x_0,y_0)(u_0) \in DF(x_0,y_0)(u_0) \quad .$$

Remark.

It is clear that

$$\text{(7)} \qquad \forall (x_0,y_0) \in \text{graph}(F),\ T_{F(x_0)}(y_0) \subset DF(x_0,y_0)(0) \quad .$$

Consequently, since the graph of $DF(x_0,y_0)$ is a convex cone, we have

(8) $$DF(x_0,y_0)(u_0) + T_{F(x_0)}(y_0) \subset DF(x_0,y_0)(u_0) \quad .$$

In particular, if F has a directional derivative, we have

(9) $$\nabla F(x_0,y_0)(u_0) + T_{F(x_0)}(y_0) \subset DF(x_0,y_0)(u_0) \quad .$$

If F is locally Lipschitzean, we obtain the equality

(10) $$T_{F(x_0)}(y_0) = DF(x_0,y_0)(0) \quad .$$

<u>Proposition</u> 2

Let x_0 and x belong to the domain of a set valued map F with closed convex graph. For any $y_0 \in F(x_0)$, we have

(11) $$F(x) - y_0 \subset DF(x_0,y_0)(x-x_0) \quad .$$ ▲

<u>Proof</u>:

Indeed, for any $h \in]0,1]$, for any $y \in F(x)$, we have $(1-h)y_0 + hy \subset F(x_0 + h(x-x_0))$. Hence $y - y_0 \in \frac{F(x_0 + h(x-x_0)) - y_0}{h}$. This implies that $y - y_0 \in DF(x_0,y_0)(x-x_0)$. ■

This inclusion allows a characterization of the minimum of a set-valued map (variational principle).

Let $P \subset Y$ be a closed convex cone of Y defining a preorder. We recall that $x_0 \in K$ achieves the minimum of set-valued map $F : K \to Y$ at $y_0 \in F(x_0)$ if

(12) $$\forall\, x \in K,\ F(x) \subset y_0 + P \quad .$$

<u>Proposition</u> 3

Let F be a set-valued map from K to Y with closed convex graph. Then $x_0 \in K$ achieves the minimum of F on K at $y_0 \in F(x_0)$ if and only if

(13) $$\forall\, u_0 \in X,\ DF(x_0,y_0)(u_0) \subset P \quad .$$

Proof:

The necessity is always true (see Proposition 4.9 of Aubin [5]). Let $x \in K$. By Proposition 2, property (13) implies

$$F(x) \subset y_0 + DF(x_0,y_0)(x-x_0) \subset y_0 + P \quad . \qquad \blacksquare$$

Let us recall the definition of a convex process G from X to Y and of its transpose.

Definition 2

A set-valued map G from X to Y is called a convex process if and only if its graph is a "closed convex cone". Its transpose is the convex process G^* from Y^* to X^* defined by

$$(14) \qquad p \in G^*(q) \Longleftrightarrow \forall\, x \in X,\ \forall\, y \in G(x),\ \langle p,x\rangle \leq \langle q,y\rangle$$

or, equivalently,

$$(15) \qquad p \in G^*(q) \text{ if and only if } (p,-q) \in [\mathrm{graph}(G)]^- \quad . \qquad \blacktriangle$$

It is clear that the bitranspose G^{**} of a convex process G is equal to G. Convex processes are the set-valued analogues of *continuous linear operators, whose graphs are closed* subspaces of $X \times Y$. If $G \in \mathcal{L}(X,Y)$, the transpose of G (regarded as a convex process) coincides with the transpose of G (regarded as a continuous linear operator).

We shall single out the transpose of $DF(x_0,y_0)$, which is a convex process.

Definition 3

We shall say that the transpose $DF(x_0,y_0)^*$ of the contingent derivative $DF(x_0,y_0)$ of a map F with closed convex graph is the "codifferential" of F at (x_0,y_0). $\blacktriangle$

It is a convex process from Y^* to X^* that associates with any $q_0 \in Y^*$ a (possibly empty) closed convex subset $DF(x_0,y_0)^*(q_0)$ of X^*. So, $p_0 \in DF(x_0,y_0)^*(q_0)$ if and only if

$$(16) \qquad \forall\, u \in X,\ \forall\, v \in DF(x_0,y_0)(u),\ \langle p_0,u\rangle \leq \langle q_0,v\rangle \quad .$$

Since the graph of F is closed and convex, we can characterize the codifferential in the following way.

Proposition 4

Let F be a set-valued map from X to Y with closed convex graph. The following statements are equivalent.

a). $p_0 \in DF(x_0,y_0)^*(q_0)$

b). $\forall\, x \in X,\ \forall\, y \in F(x),\ \langle q_0, y_0-y\rangle \leq \langle p_0, x_0-x\rangle$.

Proof:

We use the facts that the graph of $DF(x_0,y_0)$ is the tangent cone to graph(F) at (x_0,y_0) and that its negative polar cone is the normal cone to graph(F) at (x_0,y_0) (see Proposition 2.1 above). Hence $p_0 \in DF(x_0,y_0)^*(q_0)$ if and only if $(p_0,-q_0) \in N_{graph(F)}(x_0,y_0)$. This is exactly the second statement of the proposition. ■

The following property of monocity holds:

Proposition 5

Let F be a set-valued map from X to Y with closed convex graph. Then if we take $p_i \in DF(x_i,y_i)(q_i)$ $(i=1,2)$, we have

(17) $$\langle q_1 - q_2,\ y_1 - y_2\rangle \leq \langle p_1 - p_2,\ x_1 - x_2\rangle \quad . \qquad \blacktriangle$$

Proof:

It is left as an exercise.

7. Conjugate functions of set-valued maps with closed convex graph

We shall characterize F by its "conjugate function" defined on $X^* \times Y^*$ in the following way.

Definition 1

We shall say that the function f^* from $X^* \times Y^*$ to $]-\infty,+\infty]$ defined by

$$(1) \qquad f^*(p,q) = \sup_{(x,y)\in \mathrm{graph}(F)} (\langle p,x\rangle - \langle q,y\rangle) \qquad \blacktriangle$$

is the conjugate function of F.

In other words, $f^*(p,q) = \sigma_{\mathrm{graph}(F)}(p,-q)$ is, up to a minus sign, the support function of the graph. So, it is convex, lower semicontinuous and positively homogeneous. Hence $y \in F(x)$ if and only if $\forall\, p \in X^*$, $\forall\, q \in Y^*$, $\langle p,x\rangle - \langle q,y\rangle \le f^*(p,q)$.

Proposition 1

Let F be a set-valued map with closed convex graph. The following statements are equivalent

a). $p_0 \in DF(x_0,y_0)^*(q_0)$

b). $\langle p_0,x_0\rangle = \langle q_0,y_0\rangle + f^*(p_0,q_0)$. $\blacktriangle$

Proof:

Statement b) is clearly equivalent to statement b) of proposition 6.4.

Since $f^*(p,q) = \sup_{x\in X} [\langle p,x\rangle - (-\sigma(F(x),-q)]$ where $\sigma(F(x),q) \doteq \sup_{y\in F(x)} \langle p,y\rangle$ is the support function of $F(x)$, $p \to f^*(p,q)$ is the conjugate function of $x \mapsto -\sigma(F(x),-q)$. The latter is a convex function (for the graph of F is convex) and is lower semicontinuous whenever F is an upper hemicontinuous set-valued map. In this case, it is equal to its biconjugate.

Proposition 2

If F is an upper hemicontinuous map from X to Y with convex graph, we can write

$$(2) \qquad \sigma(F(x),q) = \inf_{p\in X^*} [f^*(p,-q) - \langle p,x\rangle] \ . \qquad \blacktriangle$$

Proposition 3

If G is a convex process from X to Y, its conjugate function is the indicator of the graph G^*:

$$g^*(p,q) = \begin{cases} 0 & \text{if } p \in G^*(q) \\ +\infty & \text{if } p \notin G^*(q) \end{cases} \quad . \tag{3}$$ ▲

Proof: It is left as an exercise. ■

Let F_1 and F_2 be two set-valued maps with closed convex graph. We consider the sum $F \doteq F_1 + F_2$, defined by $F(x) \doteq F_1(x) + F_2(x)$. Its graph is obviously convex, but not necessarily closed. There are several instances when this is the case. Let us mention for instance the following one:

Lemma 1

Let us assume that F_1 has a weakly closed graph and that F_2 is upper hemicontinuous with weakly compact convex images. Then the graph of $F_1 + F_2$ is also closed. ▲

Proof:

Let us consider a sequence of elements $(x_n, y_{1n} + y_{2n}) \in \text{graph}(F_1+F_2)$ converging to (x,z). Since F_2 is upper hemicontinuous, we can associate with any $p \in Y^*$ and $\varepsilon > 0$ an integer $N(\varepsilon)$ such that $\sigma(F_2(x_n),-p) \leq \sigma(F_2(x),-p) + \varepsilon/2$ and $\langle p, y_{1n} + y_{2n}\rangle \leq \langle p,z\rangle + \varepsilon/2$ when $n \geq N(\varepsilon)$. Therefore, $\langle p, y_{1n}\rangle \leq \langle p, y_{1n} + y_{2n}\rangle + \langle -p, y_{2n}\rangle \geq \langle p,z\rangle + \sigma(F_2(x),-p) + \varepsilon$ when $n \geq N(\varepsilon)$. Since $F_2(x)$ is weakly compact, the right-hand side of the latter inequality is finite and consequently,

$$\forall\, p \in Y^*, \ \sup_{n \geq 0} \langle p, y_{1n}\rangle < +\infty$$

By the uniform boundedness theorem (see for instance Aubin [2], chap. 4, 1, p.73), the sequence of elements y_{1n} is relatively weakly compact. So, subsequences of elements (x_n, y_{1n}) and (x_n, y_{2n}) converge to elements (x, y_1) and (x, y_2). Since the graphs are weakly closed (the one of F_1 by assumption, the one of F_2 since it is upper hemicontinuous), we infer that $y_1 \in F_1(x)$, $y_2 \in F_2(x)$ and $z = y_1 + y_2 \in (F_1 + F_2)(x)$. ■

Remark:

The graph of F_1 is weakly closed whenever it is convex and (strongly) closed. ■

It is clear that

(4) $$(f_1 + f_2)^*(p_1 + p_2,q) \leq f_1^*(p_1,q) + f_2^*(p_2,q) \quad .$$

We shall deduce from a minimax theorem that equality holds under reasonable assumptions.

Theorem 1

Let F_1 and F_2 be two set-valued maps with closed convex graph from X to Y. We assume that

(5) $$0 \in \mathrm{Int}(\mathrm{Dom}\, F_1 - \mathrm{Dom}\, F_2)$$

and that F_1 is upper hemicontinuous.

Then, if $(f_1 + f_2)^*(p_0,q_0)$ is finite, we can write

(6) $$(f_1 + f_2)^*(p_0,q_0) = f_1^*(p_1,q_0) + f_2^*(p_2,q_0) \quad \text{with} \quad p_0 = p_1 + p_2 \quad .$$ ▲

Proof:

Let p_0 and q_0 such that

(7) $$(f_1 + f_2)^*(p_0,q_0) = \sup_{\substack{x \in X \\ y_i \in F_i(x)(1-1,2)}} [\langle p_0,x\rangle - \langle q_0,y_1\rangle - \langle q_0,y_2\rangle] < +\infty$$

is finite.

Since F_1 is upper hemicontinuous, $x \mapsto -\sigma(F_1(x),-q_0)$ is convex and lower semicontinuous and hence, is the conjugate function of $p \mapsto f_1^*(p,q_0)$. (See Proposition 2).

Let us set $\mathcal{F}_i = \mathrm{graph}(F_i)(i=1,2)$ and

(8) $$\varphi(p;x,y_2) = F_1^*(p,q_0) - \langle p,x\rangle + \langle p_0,x\rangle - \langle q_0,y_2\rangle \quad .$$

So, we can write

(9) $$(f_1 + f_2)^*(p_0,q_0) = \sup_{(x,y_2)\in \mathcal{F}_2} \ \inf_{p \in \mathrm{Dom} f_1^*(\cdot,q_0)} \varphi(p;x,y_2) \quad .$$

The functions $p \mapsto \varphi(p;x,y_2)$ are obviously convex and lower semicontinuous and $(x,y_2) \to \varphi(p;x,y_2)$ are affine continuous. In order to apply the lopsided minimax theorem of Aubin [3], theorem 13.1.2, we have to prove that

$$(10) \quad \begin{cases} \exists \gamma > 0 \ \text{ such that, } \ \forall v \in \gamma B, \ \exists (x,y_2) \in F_2 \ \text{ such that} \\ a \doteq \sup\limits_{p \in \mathrm{Dom} f_1^*(\cdot,q_0)} [\langle p,v\rangle - \varphi(p;x,y_2)] \ \text{ is finite.} \end{cases}$$

This is the case, by assumption (5): Let $\gamma > 0$ such that $\gamma B \subset \mathrm{Dom}\, F_2 - \mathrm{Dom}\, F_1$, so there exists $x \in \mathrm{Dom}\, F_2$ such that $v + x \in \mathrm{Dom}\, F_1$. We choose $y_2 \in F_2(x)$.

Hence, since $-\sigma(F_1(\cdot),-q_0)$ is the conjugate function of $f_1^*(\cdot,q_0)$, we check that

$$(11) \qquad a \doteq -\sigma(F_1(x+v),q_0) - \langle p_0,x\rangle + \langle q_0,y_1\rangle$$

is thus finite. Hence, there exists $p_1 \in \mathrm{Dom}\, f_1^*(\cdot,q_0)$ such that

$$\begin{aligned}(f_1 + f_2)^*(p_0,q_0) &= \sup_{(x,y_2)\in F_2} \varphi(p,x,y_2) \\ &= f_1^*(p_1,q_0) + \sup_{(x,y_2)} [\langle p_0 - p_1,x\rangle - \langle q_0,y_2\rangle] \\ &= f_1^*(p_1,q_0) + f_2^*(p_0 - p_1,q_0) \ .\end{aligned}$$

Corollary 1

Let G_1 and G_2 be two convex processes from X to Y. If G_1 is upper hemicontinuous and if $0 \in \mathrm{Int}(\mathrm{Dom}\, G_1 - \mathrm{Dom}\, G_2)$, then

$$(G_1 + G_2)^* = G_1^* + G_2^* \ . \qquad \blacktriangle$$

Proof:

By Theorem 1, when $p \in (G_1 + G_2)^*(q)$, we can associate p_1 and p_2 such that $p = p_1 + p_2$ and $(g_1 + g_2)^*(p,q) = g_1^*(p_1,q) + g_2^*(p_2,q)$.

Each term of the above equality can be either 0 or ∞. Since $p \in (G_1 + G_2)^*(q)$, they are all equal to 0. But to say that $g_1^*(p_i,q) = 0$ means that $p_i \in G_i^*(q)\,(i=1,2)$. Hence

$(G_1 + G_2)^*(q) \subset G_1^*(q) + G_2^*(q)$. The converse inclusion being obvious, equality $(G_1 + G_2)^*(q) = G_1^*(q) + G_2^*(q)$ ensues.

Let F be a set-valued map from X to Y with closed convex graph and $A \in \mathcal{L}(Z,Y)$. We shall compute the conjugate function of the set-valued map $G \doteq FA$ from Z to Y. We always obtain the inequality: if $q_0 \in Y^*$ and $p_0 \in Z^*$, then

$$\forall\, p \in X^* \text{ such that } A^*p = p_0,\ g^*(p_0,q_0) \leq f^*(p,q_0) \quad . \tag{12}$$

Under reasonable assumptions, we shall prove that equality holds for some p satisfying $A^*p = p_0$.

<u>Theorem 2</u>

Let F be an upper hemicontinuous set-valued map with convex graph from X to Y and $A \in \mathcal{L}(Z,Y)$. We posit the following condition

$$0 \in \mathrm{Int}(\mathrm{Im}\, A - \mathrm{Dom}\, F) \quad . \tag{13}$$

Let $G \doteq FA$ and $q_0 \in X^*$, $p_0 \in Z^*$ such that $g^*(p_0,q_0)$ is finite. Then

$$\exists\, \bar{p} \in X^* \text{ such that } A^*\bar{p} = p_0 \text{ and } g^*(p_0,q_0) = f^*(\bar{p},q_0) \quad . \tag{14}$$

<u>Remark</u>

Note that assumption (13) is satisfied whenever A is surjective or $\mathrm{Im}\, A \cap \mathrm{Int}\,\mathrm{Dom}\, F \neq \emptyset$.

<u>Proof</u>:

Since F is upper hemicontinuous, Proposition 2 implies that $\sigma(F(Az),-q_0) = \inf_{p\in X^*} [f^*(p,q_0) - \langle p,Az\rangle]$. Hence

$$g^*(p_0,q_0) \doteq \sup_{\substack{z\in Z\\ y\in F(Az)}} [\langle p_0,z\rangle - \langle q_0,y\rangle]$$

$$= \sup_{z\in Z} [\langle p_0,z\rangle + \sigma(F(Az),-q_0)]$$

$$= \sup_{z\in Z} \inf_{p\in X^*} [\langle p_0 - A^*p,z\rangle + f^*(p,q_0)] \quad .$$

The function φ defined on $\mathrm{Dom}\, f^*(\cdot,q_0) \times Z$ by

$$\varphi(p,z) = f^*(p,q_0) + \langle p_0 - A^* p, z\rangle \tag{15}$$

is convex and lower semicontinuous with respect to p and affine and continuous with respect to z. By assumption (13), there exists $\gamma > 0$ such that, $\forall\, v \in \gamma B$, there exists $z \in Z$ satisfying $v + Az \in \mathrm{Dom}\, F$. Consequently,

$$\begin{cases} a \doteq \sup\limits_{p \in \mathrm{Dom}\, f^*(\cdot,q_0)} [\langle p,v\rangle - \varphi(p,z)] \\ \quad = -\sigma(F(v+Az),-p) - \langle p_0,z\rangle \end{cases} \tag{16}$$

if finite. Therefore, we can apply the minimax theorem of Aubin [3], (theorem 13.1.2): There exists $\bar{p} \in X^*$ such that

$$\begin{aligned} g^*(p_0,q_0) &= \sup_{z \in Z} [\langle p_0 - A^*\bar{p}, z\rangle + f^*(\bar{p},q_0)] \\ &= f^*(\bar{p},q_0) \quad \text{where } p_0 = A^*\bar{p}\ . \end{aligned}$$

■

Corollary 2

Let F be an upper hemicontinuous convex process from X to Y and $A \in \mathcal{L}(Z,Y)$. We posit the following condition

$$0 \in \mathrm{Int}(\mathrm{Im}\, A - \mathrm{Dom}\, F)\ . \tag{17}$$

Then the formula

$$(FA)^* = A^* F^* \tag{18}$$

holds true. ▲

Proof:

The inclusion $A^*F^* \subset (FA)^*$ is obvious. For proving the opposite inclusion, we set $G \doteq FA$ and we take $p_0 \in G^*(q_0)$. Since G is a convex process, this means that $g^*(p_0,q_0) = 0$. By theorem 2, whose assumptions are satisfied, we deduce that there exists $\bar{p}$ such that $A^*\bar{p} = p_0$ and $g^*(p_0,q_0) = f^*(\bar{p},q_0)$. Hence $f^*(\bar{p},q_0) = 0$, i.e., $\bar{p} \in F^*(q_0)$. Hence $p_0 = A^*\bar{p} \in A^*F^*(q_0)$. ■

8. Calculus on derivatives and codifferentials

We begin by proving the "chain rule".

Theorem 1

Let F be a set-valued map from X to Y with closed convex graph and $A \in \mathcal{L}(Z,Y)$. We assume that

$$(1) \qquad 0 \in \mathrm{Int}(\mathrm{Im}\, A - \mathrm{Dom}\, F) \; .$$

Then the following chain rule formulas hold

$$(2) \quad \begin{cases} \text{i)} & D(FA)(z_0,y_0) = DF(Az_0,y_0)A \\ \text{ii)} & D(FA)(z_0,y_0)^* = A^* DF(Az_0,y_0)^* \; . \end{cases}$$ ▲

Proof:

We know that the graph $\mathcal{G}$ of $G \doteq FA$, which is closed and convex, is equal to $(A \times 1)^{-1}\mathcal{F}$, where $\mathcal{F}$ is the graph of F. The assumption $0 \in \mathrm{Int}(\mathrm{Im}\, A - \mathrm{Dom}\, F)$ implies obviously that, in $X \times Y$, $0 \in \mathrm{Int}(\mathrm{Im}(A \times 1) - \mathcal{F})$. So, by theorem 5.1, we know that $T_{\mathcal{G}}(z_0,y_0) = (A \times 1)^{-1} T_{\mathcal{F}}(Az_0,y_0)$ and, by proposition 5.4, that $N_{\mathcal{G}}(z_0,y_0) = (A^* \times 1) N_{\mathcal{F}}(Az_0,y_0)$. This implies formulas (2) i) and ii). Note that <u>assumption (1) is always satisfied when A is surjective</u> or <u>when $\mathrm{Im}\, A \cap \mathrm{Int}\, \mathrm{Dom}\, F \neq \emptyset$</u>. ■

Let us still consider a set-valued map F with closed convex graph from X to Y and $A \in \mathcal{L}(Y,Z)$. The graph of the set-valued map AF from X to Z is still convex, but not necessarily closed. We shall denote by $\overline{AF}$ the set-valued map whose graph is the closure of the graph of AF.

Theorem 2

Let F be a set-valued map from X to Y with closed convex graph and $A \in \mathcal{L}(Y,Z)$. Then

$$(3) \quad \begin{cases} \text{i)} & D(\overline{AF})(x_0,Ay_0) = \overline{A.\, DF(x_0,y_0)} \\ \text{ii)} & D(\overline{AF})(x_0,Ay_0)^* = DF(x_0,y_0)^* A^* \; . \end{cases}$$ ▲

Proof:

We note that the graph $\mathcal{G}$ of $G \doteq AF$ is $(1 \times A)\mathcal{F}$, where $\mathcal{F}$ is the graph F.

We deduce from Proposition 5.3 that $T_{\mathcal{G}}(x_0, Ay_0) = c\ell(1 \times A)T_{\mathcal{F}}(x_0, y_0)$ and that $N_{\mathcal{G}}(x_0, Ay_0) = (1 \times A)^* N_{\mathcal{F}}(x_0, y_0)$. So formulas (3) i) and ii) ensue. ∎

We now compute the derivative of an intersection.

Proposition 1

Let us consider $n + 1$ set-valued maps F_i with closed convex graph and let F be the set-valued map from X to Y defined by

$$(4) \qquad F(x) \doteq \bigcap_{i=0}^{n} F_i(x) \quad .$$

Let $I(x_0, y_0)$ denote the set of indexes i such that $(x_0, y_0) \in \text{Bound}(\text{graph}(F_i))$. If we assume that

$$(5) \qquad \begin{cases} \exists\, \gamma > 0 \text{ such that, } \forall\, i=1,\dots,n,\ \forall\, u \in \gamma B,\ \exists\, x_0 \in \text{Dom } F_0 \\ \text{and } y_0 \in F(x_0) \text{ such that } y_0 \in F_i(x_0 + u_i) - v_i \ (i=1,\dots,n) \end{cases}$$

then,

$$(6) \qquad DF(x_0, y_0) = \bigcap_{i \in J(x_0, y_0)} DF_i(x_0, y_0) \quad . \qquad \blacktriangle$$

Proof:

The graph $\mathcal{F}$ of F is the intersection of the $n + 1$ graphs $\mathcal{F}_i$ of the maps F_i. It is easy to check that assumption (5) implies that for any $i=1,\dots,n$, for any $(u_i, v_i) \in \gamma(B \times B)$, there exists $(x_0, y_0) \in \mathcal{F}_0$ such that $(x_0 + u_i, y_0 + v_i) \in \mathcal{F}_i$. Thus, by Proposition 5.5, $T_{\mathcal{F}}(x_0, y_0) = \bigcap_{i=0}^{n} T_{\mathcal{F}_i}(x_0, y_0) = \bigcap_{i \in J(x_0, y_0)} T_{\mathcal{F}_i}(x_0, y_0)$ (for $T_{\mathcal{F}_i}(x_0, y_0) = X \times Y$ when $i \notin J(x_0, y_0)$.). This means that for all $u_0 \in X$, $DF(x_0, y_0)(u_0) = \bigcap_{i \in J(x_0, y_0)} DF_i(x_0, y_0)$. ∎

For a calculus on set-valued maps to be useful, we need a formula on the derivative of the sum F of two set-valued maps F_1 and F_2, defined by

$$F(x) = F_1(x) + F_2(x) \quad .$$

We begin by noticing that the graph of F is not necessarily closed when the graphs of $F_i (i=1,2)$ are closed. We recall that it is the case, though, when one of the set-valued maps is upper hemicontinuous with compact convex images.

There are instances when we can compare the derivatives of $F_1 + F_2$ and of F_1 and F_2. For example, it is easy to show that if the maps F_i are locally Lipschitzean with convex graph, then $D(F_1 + F_2)(x_0, y_1 + y_2) \subset DF_1(x_0,y_1) + DF_2(x_0,y_2)$.

The situation is much better for codifferentials: we always obtain the inclusion

(7) $$DF_1(x_0,y_1)^* + DF_2(x_0,y_2)^* \subset D(F_1 + F_2)^*(x_0, y_1 + y_2) \quad .$$

Indeed, if $p_i \in DF_i(x_0,y_i)^*(q)$, then equalities $\langle p_i,x_0\rangle = f_i^*(p_i,q) + \langle q,y_i\rangle (i=1,2)$ imply inequality $\langle p_1 + p_2, x_0\rangle = f_1^*(p_1,q) + f_2^*(p_2,q) + \langle q, y_1 + y_2\rangle \geq (f_1 + f_2)^*(p_1 + p_2, q) + \langle q, y_1 + y_2\rangle$. Since the other inequality is always true, we infer that $\langle p_1 + p_2, x_0\rangle = (f_1 + f_2)^*(p_1 + p_2, q) + \langle q, y_1 + y_2\rangle$, i.e., that $p_1 + p_2 \in D(F_1 + F_2)^*(x_0, y_1 + y_2)$.

■

We now state that under reasonable assumptions the codifferential of the sum of two set-valued maps $F_i(i=1,2)$ is equal to the sum of the codifferentials.

<u>Theorem</u> 3

Let F_1 and F_2 be two set-valued maps with closed convex graph from X to Y. We assume that

(8) $$0 \in \mathrm{Int}(\mathrm{Dom}\, F_1 - \mathrm{Dom}\, F_2) \quad .$$

and that F_1 is upper hemicontinuous with (weakly) compact values. Then, for all $x_0 \in \mathrm{Dom}\, F_1 \cap \mathrm{Dom}\, F_2$, $y_1 \in F_1(x_0)$, $y_2 \in F_2(x_0)$, we have:

(9) $$D(F_1 + F_2)(x_0, y_1 + y_2)^* = DF_1(x_0,y_1)^* + DF_2(x_0,y_2)^* \quad . \quad \blacktriangle$$

This formula justifies the usefulness of the concept of codifferential.

Proof of Theorem 3

Assume that $p_0 \in D(F_1 + F_2)(x_0, y_1 + y_2)^*(q_0)$. So, $\langle p_0, x_0\rangle = (f_1 + f_2)^*(p_0, q_0) + \langle q_0, y_1 + y_2\rangle$. Theorem 7.1 implies the existence of p_1 such that

$$0 = (f_1^*(p_1, q_0) + \langle q_0, y_1\rangle - \langle p_1, x_0\rangle) +$$
$$+ (f_2^*(p_0 - p_1, q_0) + \langle q_0, y_2\rangle - \langle p_0 - p_1, x_0\rangle) .$$

Since the two terms of this sum are nonnegative, then each of these two terms is equal to zero. This means that $p_1 \in DF_1(x_0, y_1)^*(q_0)$ and $p_0 - p_1 \in DF_2(x_0, y_2)^*(q_0)$. So $D(F_1 + F_2)(x_0, y_1 + y_2)^*(q_0) \subset DF_1(x_0, y_1)^*(q_0) + DF_2(x_0, y_2)^*(q_0)$.

Since the other inclusion (7) is true, equality (9) holds. ■

We state now two consequences. Let us recall that we denote by ψ_K the indicator of K, which is the set valued map from X to Y defined by

$$\psi_K(x) = \begin{cases} \{0\} & \text{if } x \in K \\ \emptyset & \text{if } x \notin K \end{cases} . \tag{10}$$

We recall that

$$D\psi_K(x) = \psi_{T_K(x)} . \tag{11}$$

The restriction of a set-valued map F from X to Y is the sum $F + \psi_K$.

We can compute the derivative of the restriction of a set-valued map with closed convex graph.

Proposition 2

Let F be a set-valued map from X to Y with closed convex graph and $F + \psi_K$ be its restriction to the closed convex subset $K \subset X$. If

$$0 \in \mathrm{Int}(\mathrm{Dom}\, F - K) \tag{12}$$

then for all $x \in K$ and $y \in F(x)$,

$$(13) \begin{cases} \text{i)} & D(F + \psi_K)(x,y)^*(q) = DF(x,y)^*(q) + N_K(x) \\ \text{ii)} & D(F + \psi_K)(x,y) = DF(x,y) + \psi_{T_K(x)} \text{ is the restriction of } DF(x,y) \text{ to } T_K(x) \end{cases}$$ ▲

Proof:

The indicator ψ_K is an upper hemicontinuous set-valued map from X to Y whose graph $K \times \{0\}$ is a closed convex subset. The normal cone to $K \times \{0\}$ is $N_K(x) \times Y^*$. Then, for any $q \in Y^*$, $p \in D\psi_K(x,0)^*(q)$ if and only if $p \in N_K(x)$.

We apply Theorem 3 and we obtain formula (13) i). We deduce formula (13) ii) by transposition. ∎

Proposition 3

Let F be a set-valued map from X to Y with closed convex graph and $P \subset Y$ be a closed convex cone. We assume that the subsets $F_+(x) = F(x) + P$ are closed for all x. Then $\forall x \in \mathrm{Dom}\, F$, $\forall y \in F(x)$,

$$DF_+(x,y)^*(q) = \begin{cases} DF(x,y)^*(q) & \text{if } q \in P^+ \\ \emptyset & \text{if } q \notin P^+ . \end{cases} \tag{14}$$

($DF_+(x,y)^*$ is the restriction of $DF(x,y)^*$ to P^+.)

Proof:

We consider the constant set-valued map $G : x \mapsto P$ whose graph is $X \times P$. Its normal cone to $(x,0)$ is $\{0\} \times P^-$. Hence $DG(x,0)^*$ is the indicator of the positive polar cone P^+. Since G is upper hemicontinuous and $\mathrm{Dom}\, G = X$, Theorem 3 implies formula (14). ∎

9. Upper derivatives and codifferentials of convex operators

Single valued maps whose graphs are convex (resp. closed and convex) are the affine operators (resp. continuous and affine). The latter statement follows from the closed graph theorem (see for instance Aubin [2], chap. 4, §3, p. 83).

We shall investigate now the properties of P - convex and P - lower semicontinuous operators.

Definition 1

Let P be a closed convex cone of Y and A be a single valued map from a convex subset K to Y, which we extend to X by setting $A(x) = \emptyset$ when $x \notin K$.

We say that A is P - convex if, for all convex combinations $\sum_{i=1}^{n} \lambda^i x_i$ of elements of X we have

$$\sum_{i=1}^{n} \lambda^i A(x_i) \in A(\sum \lambda^i A(x_i)) + P \quad . \tag{1}$$

We say that A is P - lower semicontinuous (in short, P - ℓ.s.c.) if

$$\forall p \in Y^*, \forall \varepsilon > 0, \exists N(x_0) \text{ such that } A(x) \in A(x_0) + P + B(p,\varepsilon) \quad . \tag{2}$$ ▲

Let us set $A_+(x) \doteq A(x) + P$ when $x \in K$ and $A_+(x) = \emptyset$ when $x \notin K$. To say that A is P - convex amounts to saying that the graph of F is convex and to say that A is P - ℓ.s.c. amount to saying that the set-valued map A_+ is upper hemicontinuous (and thus, has a closed graph). ■

Definition 2

We shall say that the convex process $D_+A(x)$ defined by

$$D_+A(x)(u) \doteq DA_+(x,A(x))(u) \tag{3}$$

is the <u>upper contingent derivative</u> of <u>A at x</u> and that its transpose

$$D_+A(x)^*(q) \doteq DA_+(x,A(x))^*(q) \tag{4}$$

is the <u>upper codifferential</u> of A at x.

We shall say that the <u>conjugate function</u> of the set-valued map A_+ is, more simply, the conjugate function of the P - convex and P - ℓ.s.c. operator A. ▲

Proposition 1

The conjugate function of a P - convex and P - ℓ.s.c. operator A is equal to

$$a^*(p_0,q_0) = \begin{cases} \sup\limits_{x\in K} [\langle p_0,x\rangle - \langle q_0,A(x)\rangle] & \text{when } q_0 \in P^+ \\ +\infty & \text{when } q_0 \notin P^+ . \end{cases} \tag{5}$$

The domains of the upper codifferentials $D_+A(x)^*$ are contained in P^+. If $q_0 \in P^+$, the following statements are equivalent:

a) $p_0 \in D_+A(x_0)^*(q_0)$

b) $\langle p_0,x_0\rangle = a^*(p_0,q_0) + \langle q_0,A(x_0)\rangle$

c) $\forall\, x \in X,\ \langle q_0, A(x_0) - A(x)\rangle \le \langle p_0, x_0 - x\rangle$

d) $\forall\, u \in X,\ \forall\, v \in D_+A(x_0)(u),\ \langle p_0,u\rangle \le \langle q_0,v\rangle$. ▲

Proof:

This follows from the fact that the support function $\sigma(P,q) = 0$ when $q \in -P^+$ and $\sigma(P,q) = \infty$ when $q \notin -P^+$. Therefore,

$$\begin{aligned} a^*(p,q) &= \sup_{\substack{x\in X\\ z\in P}} [\langle p,x\rangle - \langle q,A(x)\rangle - \langle q,z\rangle] \\ &= \sup_{x\in X} [\langle p,x\rangle - \langle q,A(x)\rangle] + \sigma(P,-q) . \end{aligned}$$

The characterization of $D_+A(x)^*$ follows from Propositions 6.4 and 7.7 by using the same remark. ■

We translate now the properties of the set-valued map A_+ in terms of the P - ℓ.s.c. and convex operator A.

Theorem 1

Let A be a P - convex and P - ℓ.s.c. operator A from K to Y.

a) for any $x_0, x \in K$, we have

$$A(x) - A(x_0) \in D_+A(x_0)(x - x_0) + P \tag{6}$$

b) x_0 minimizes A on K, in the sense that $A(x) - A(x_0) \in P$ for all $x \in K$, if and only if

$$\forall u \in \text{Dom } D_+A(x_0),\ D_+A(x_0)(u) \in P \tag{7}$$

c) If $B \in \mathcal{L}(Z,X)$ and if

$$0 \in \text{Int}(\text{Im}(B) - \text{Dom } A) \tag{8}$$

then

$$\begin{cases} \text{i)} & D_+(AB)(z_0) = (D_+A)(Bz_0)B \\ \text{ii)} & D_+(AB)^*(z_0) = B^* D_+A(Bz_0)^* \end{cases} \tag{9}$$

d) If A_1 and A_2 are two P - convex and P - ℓ.s.c. operators satisfying

$$0 \in \text{Int}(\text{Dom } A_1 - \text{Dom } A_2) \tag{10}$$

then

$$D_+(A_1 + A_2)(x)^* = D_+A_1(x)^* + D_+A_2(x)^* \tag{11}$$

e) Let us assume that

$$0 \in \text{Int}(K - \text{Dom } A) \tag{12}$$

If A_K denotes the restriction of A to K, we have

$$\begin{cases} \text{i)} & D_+A_K(x)^* = D_+A(x)^* + N_K(x) \\ \text{ii)} & D_+A_K(x) \text{ is the restriction of } D_+A(x) \text{ to } T_K(x) \end{cases} \tag{13}$$

▲

These statements follows obviously from Propositions 6.2, 6.3, Theorem 8.1, Proposition 8.1, Theorem 8.3 and Proposition 8.2 respectively. ■

10. Upper derivatives and subdifferentials of convex functions

Let V be a real-valued function defined on a convex subset K of X. We extend it to X by setting

$$(1)\qquad V(x) \doteq \infty \quad \text{when} \quad x \notin K \quad (\text{we say that } K = \mathrm{Dom}\, V) \; .$$

We associate to V the set-valued map from X to $\mathbb{R}$ defined by

$$(2)\qquad V_+(x) \doteq V(x) + \mathbb{R}_+ \ \text{ if } \ x \in K, \ V_+(x) = \emptyset \ \text{ if } \ x \notin K \; ,$$

whose graph is the epigraph of V.

The function V is convex if and only if V_+ has a convex graph and V is lower semicontinuous if and only if V_+ is upper hemicontinuous. This is also equivalent to the fact that the graph of V_+ is closed. Then we can define the upper contingent derivative $D_+V(x)$ by setting

$$(3)\qquad D_+V(x)(u) \doteq DV_+(u,V(x))(u) \; .$$

Since $D_+V(x)(u)$ is a half-line, we *shall identify* $D_+V(x)(u)$ with the origin of this half-line. So, for real-valued functions, we actually set

$$(4)\qquad D_+V(x)(u) \doteq \inf\{v \mid v \in DV_+(x,V(x))(u) \; .$$

As in Theorem 5.1 of Aubin [5], we can check that

$$(5)\qquad D_+V(x_0)(u_0) = \liminf_{u \to u_0} \lim_{h \to 0+} \frac{V(x_0 + hu) - V(x_0)}{h} \; .$$

We note that the convexity of V implies that

$$(6)\qquad \lim_{h \to 0+} \frac{V(x_0 + hu) - V(x_0)}{h} = \inf_{h>0} \frac{V(x_0 + hu) - V(x_0)}{h} \; .$$

Also, a convex function V is continuous at x_0 if and only if it is Lipschitzean in a neighborhood of x_0. Then, in this case,

$$(7)\qquad D_+V(x_0)(u_0) = \lim_{h \to 0+} \frac{V(x_0 + hu_0) - V(x_0)}{h} \; .$$

The transposed of the contingent derivative $DV_+(x,V(x))$ is a convex process from $\mathbb{R}$ to X^*, whose domain is contained in $\mathbb{R}_+$ (by Proposition 9.1). Then, for any $q \geq 0$ we have

$$DV_+(x,V(x))^*(q) = q\, DV_+(x,V(x))^*(1) \quad .$$

It is easy to remark that

$$DV_+(x,V(x))^*(1) = \partial V(x) \tag{8}$$

is the <u>subdifferential</u> of V at x.

By Proposition 9.1, the conjugate function is infinite when $q < 0$. So we have, if $q \geq 0$

$$v^*(p_0,q_0) = \sup_{x \in K} [\langle p_0,x\rangle - q_0 V(x)] \quad .$$

We see that for $q_0 = 1$,

$$v^*(p_0,1) = V^*(p_0) \tag{9}$$

where V^* is the conjugate function in the usual sense. Therefore, when V is convex and lower semicontinuous, the following statements are equivalent:

a) $p_0 \in \partial V(x_0)$

b) $\langle p_0,x_0\rangle = V^*(p_0) + V(x_0)$

c) $\forall\, x \in X,\ V(x_0) - V(x) \leq \langle p_0,\ x_0 - x\rangle$

d) $\forall\, u \in X,\ \langle p_0,u\rangle \leq D_+V(x_0)(u)$.

(The latter statement makes explicit the fact that $p_0 \in DV_+(x_0,V(x_0))^*(1)$, i.e., the fact that for all $u \in X$ and for all $v \in DV_+(x_0,v_0)(u) = [D_+V(x_0)(u), \infty[$, $\langle p,u\rangle \leq 1.\ v$, i.e., the fact that for all $u \in X$, $\langle p,u\rangle \leq D_+V(x_0)(u)$.).

<u>Theorem</u> 9.1 on P - convex operators implies the following properties (see Rockafellar [1] for instance).

Theorem 1

Let V be a convex lower semicontinuous function from X to $]-\infty,+\infty]$ whose domain is nonempty

a) for any $x_0, x \in K$, we have

$$V(x) - V(x_0) \geq D_+V(x_0)(x - x_0) \tag{10}$$

b) x_0 minimizes V on K if an only if

$$\forall u \in \text{Dom } D_+V(x_0),\ 0 \leq D_+V(x_0)(u) \tag{11}$$

c) if $B \in \mathcal{L}(Z,X)$ and if

$$0 \in \text{Int}(\text{Im}(B) - \text{Dom } V) \quad , \tag{12}$$

then

$$\begin{cases} \text{i)} & D_+(VB)(z_0)(u) = (D_+V)(Bz_0)(Bu) \\ \text{ii)} & \partial(VB)(z_0) = B^*\partial V(Bz_0) \quad . \end{cases} \tag{13}$$

d) If W is another convex lower semicontinuous function such that

$$0 \in \text{Int}(\text{Dom } V - \text{Dom } W) \quad , \tag{14}$$

then

$$\partial(V+W)(x_0) = \partial V(x_0) + \partial W(x_0) \quad . \tag{15}$$

e) Let us assume that

$$0 \in \text{Int}(K - \text{Dom } V) \quad . \tag{16}$$

If V_K denotes the restriction of V to K, we have

$$\begin{cases} \text{i)} & D_+V_K(x)(v) = D_+V(x)(v) \quad \forall v \in T_K(u) \\ \text{ii)} & \partial V_K(x) = \partial V(x) + N_K(x) \quad . \end{cases} \tag{17}$$

▲

We mention also the following statement.

Proposition 2

Let us consider n convex lower semicontinuous functions V_i from X to $]-\infty,+\infty]$. Let V be defined by

$$V(x) \doteq \max_{i=1,..,n} V_i(x) \tag{18}$$

and

$$J(x) \doteq \{i=1,\dots,n \mid V_i(x) = V(x)\} \ . \tag{19}$$

Let us assume that the n functions V_i are continuous at x_0. Then

$$\begin{cases} \text{i)} \ \ DV(x_0)(v) = \max\limits_{i\in J(x_0)} DV_i(x_0)(v) \\ \text{ii)} \ \ \partial V(x_0) = \mathrm{co}\Big(\bigcup\limits_{i\in J(x_0)} \partial V_i(x_0)\Big) \ . \end{cases} \tag{20}$$

Proof:

We see that $V_+(x) = \bigcap\limits_{i=1}^{n} V_{i+}(x)$. So, we apply Proposition 8.1. Since $x_0 \in \bigcap\limits_{i=1}^{n} \mathrm{Int\,Dom}\, V_i$, then $J(x_0) = \{i=1,\dots,n \mid (x_0,y_0) \in \mathrm{Bound}(\mathrm{graph}\ V_{i+})\}$. Also, since the functions V_i are continuous, there exists γ such that $V_i(x_0+u) \le V_i(x_0) + \gamma$ when $u \in \gamma B(i=1,\dots,n)$. Set $y_0 \doteq V(x_0) + 2\gamma$. Then, if $u_i \in \gamma B$ and $|v_i| \le \gamma$, we have $y_0 \in V_i + (x_0+u_i) - v_i$ for $i=1,\dots,n$. Therefore, Proposition 8.1 implies that $D_+V(x_0)(v) = \bigcap\limits_{i\in J(x_0)} D_+V_i(x_0)(v)$. ■

REFERENCES

J. P. Aubin. [1] Applied Abstract Analysis. Wiley Interscience 1977.

[2] Applied Functional Analysis. Wiley Interscience 1979.

[3] Mathematical Methods of Game and Economic Theory. North Holland 1979.

[4] Cônes tangents à un sous-ensemble convexe fermé. Ann. Sc. Math. Québec. (1979).

[5] Contingent derivatives of set-valued maps and existence of solutions to nonlinear inclusions. MRC Technical Summary Report, 1979.

G. Haddad. [1] Tangential conditions for existence theorems in differential inclusions and functional differential inclusions with memory. Cah. Math. Dec. #7916. Un. Paris-Dauphine.

S. M. Robinson. [1] Regularity and Stability for convex multivalued functions. Math. Op. Res. 1(1976) 130-145.

R. T. Rockafellar. [1] Convex Analysis. Princeton University Press (1970).

[2] Monotone processes of convex and concave type. Am. Math. Soc. Memoir #77 (1967).

[3] Clarke tangent cones and the boundaries of closed sets in $\mathbb{R}^n$. (To appear)

[4] Generalized directional derivatives and subgradients of non convex functions.

[5] Directionally Lipschitzean functions and sub differential calculus. Bull. Lond. Math. Soc.

[6] La Théorie des sous-gradients et ses Applications à l'optimization. Univ. Montréal.

C. Ursescu. [1] Multifunctions with closed convex graph, Czecos. Math. J. 25(1975) 438-441.

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We complete the study of contingent cones to a subset K and of the contingent derivatives of a set valued map F when K and the graph of F are closed and convex. In this case, the contingent cone is a closed convex cone (called the tangent cone) and the contingent derivative is a convex process (a set-valued map whose graph is a closed convex cone). The transpose of the contingent derivative is another convex process, called the codifferential, which plays a quite important role.

We present a calculus of tangent cones to closed convex sets and we adopt the

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ABSTRACT (continued)

basic results of convex analysis to the case of set-valued maps with closed convex graph.

This study is motivated by the crucial role played by convex cones in optimization, fixed-point theory and flow-invariance of dynamical systems.